

Crossover from directed percolation to mean field behavior in the diffusive contact process

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Abstract. Recently Dantas, Oliveira and Stilck [J. Stat. Mech. (2007) P08009] studied how the one-dimensional diffusive contact process crosses over from the critical behavior of directed percolation to an effective mean field behaviour when the diffusion rate is sent to infinity. They showed that this crossover can be described in terms of a crossover exponent ϕ , finding the boundaries $3 \leq \phi \leq 4$ in one spatial dimension. In the present work we refine and extend this result up to four spatial dimensions by a field-theoretic calculation and extensive numerical simulations.

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1. Introduction

In non-equilibrium statistical mechanics, the study of phase transition from fluctuating phases into absorbing states continues to be a very active field [1]. One of the most important universality classes of such transitions is directed percolation (DP) [2–5], which is well understood and can be described in terms of a renormalizable field theory [6, 7]. Although this universality class plays a paradigmatic role as the Ising class in equilibrium statistical mechanics, it was primarily of theoretical interest since experimental realizations seemed to be very difficult [8]. However, very recently Takeuchi *et al.* [9] performed an experiment on the basis of turbulent liquid crystals, where the critical exponents of directed percolation could be measured.

Directed percolation is most easily introduced as a spreading process of diffusing particles on a lattice which multiply ($A \rightarrow 2A$) and self-annihilate ($A \rightarrow \emptyset$). Moreover, the density of particles is effectively limited by an exclusion principle. One of the most important models with random-sequential updates is the contact process [1] which is controlled by the rate for offspring production λ with a critical point λ_c . The DP universality class is characterized by three critical exponents $(\beta, \nu_{\parallel}, \nu_{\perp})$ which depend only on the dimensionality of the system. In $d = 1, 2, 3$ spatial dimensions the exponents

take non-trivial, probably irrational values. In $d > 4$ dimensions, however, the mean-field exponents $\beta = \nu_{\parallel} = 1$ and $\nu_{\perp} = 1/2$ become exact. Right at the upper critical dimension $d_c = 4$ there are additional logarithmic corrections.

The crossover to mean field behavior for $d > 4$ is related to the fact that the diffuse mixing becomes more efficient in high dimensions, suppressing fluctuation effects. An alternative way to enhance diffusive mixing, on which we will focus in the present work, is to increase the diffusion constant D in a low-dimensional system. In most models such as the contact process, diffusion is effectively included by the circumstance that a particle creates offspring at an empty randomly chosen nearest-neighbour site. In such models the effective diffusion rate is fixed but it is straight forward to add explicit diffusion of solitary particles so that the diffusion rate can be controlled.

By varying the diffusion constant in the contact process, one expects the following phenomenology. Clearly, in the limit $D \rightarrow \infty$, where all sites are mutually coupled, mean field theory becomes exact. For a large but finite diffusion constant, however, one expects a crossover at a typical length scale $\xi_{\perp}^{(c)}$ and an associated typical time scale $\xi_{\parallel}^{(c)}$ which grow with D . Below these scales, the process exhibits an effective mean-field behavior while it crosses over to DP on scales larger than $\xi_{\perp}^{(c)}$ and $\xi_{\parallel}^{(c)}$. In addition, the critical rate for offspring production, $\lambda_c(D)$, decreases with D and reaches the analytically known mean field value $\lambda_c(\infty)$ in the limit $D \rightarrow \infty$.

The influence of the diffusion rate on λ_c was first studied by Konno [10] from a mathematical point of view. He pointed out that the critical threshold $\lambda_c(D)$ approaches the asymptotic value $\lambda_c(\infty)$ algebraically as

$$\lambda_c(D) - \lambda_c(\infty) \sim D^{-1/\phi}, \quad (1)$$

where the crossover exponent $\phi = 3$ in one, $\phi = 1 + \log. \text{ corrections}$ in two and $\phi = 1$ in more than two spatial dimensions. More recently, by series expansion and partial differential approximants, Dantas *et al.* [11] found the boundaries $3 \leq \phi \leq 4$ for this exponent. In the present work we describe a pure field theoretic approach which allows an easy calculation of the crossover exponents. Furthermore we present numerical results for one to four spatial dimensions.

2. Field-theoretical approach

The diffusive DP model is most easily described by a contact process defined by the three micro processes (i) creation of a particle on, (ii) hopping to one of the neighboring places, and (iii) death of a particle which take place with rates λ , D and 1 respectively. These dynamic laws result in the coarse grained Langevin equation [12]

$$\frac{\partial}{\partial t} \rho(\vec{x}, t) = \frac{D}{2d} \Delta \rho(\vec{x}, t) + (\lambda - 1) \rho(\vec{x}, t) - \lambda \rho^2(\vec{x}, t) + \xi(\vec{x}, t) \quad (2)$$

with spatial dimension d , density of particles $\rho(\vec{x}, t)$ and multiplicative noise $\xi(\vec{x}, t)$ with the correlations

$$\langle \xi(\vec{x}, t) \xi(\vec{y}, t') \rangle \sim \rho(\vec{x}, t) \delta^d(\vec{x} - \vec{y}) \delta(t - t'). \quad (3)$$

Based on this equation one can set up a field theory by eliminating the noise and introducing a response field. After appropriate rescaling of fields and parameters, the field theoretic action reads (see [7] and references therein)

$$\mathcal{S} = \int d^d x dt \frac{1}{2} \left(\tilde{\phi} \frac{\partial}{\partial t} \phi - \phi \frac{\partial}{\partial t} \tilde{\phi} \right) + \frac{D}{2d} (\nabla \tilde{\phi}) (\nabla \phi) - \tilde{\phi} \kappa \phi + g (\tilde{\phi} \phi^2 - \tilde{\phi}^2 \phi) \quad (4)$$

with density (response) fields ϕ ($\tilde{\phi}$), control parameter $\kappa = \lambda - 1$ and coupling g . As seen from (4) the field theory of DP involves a directed propagator and two cubic interaction vertexes with the same coupling strength g . Using these Feynman rules, one can calculate the next order (one loop) corrections to propagator and vertexes. For instance, the inverse propagator to one loop order is given by

$$\Gamma^{(1,1)}(k, \omega) = \text{---} \blacktriangleright \text{---} - \text{---} \blacktriangleleft \text{---} + \mathcal{O}(g^4). \quad (5)$$

Naturally these loop corrections lead to divergent contributions, which receive a physical meaning upon applying a regularization and renormalization procedure. The divergences then appear as corrections to the meanfield scaling exponents

$$\nu_{\perp} = \frac{1}{2} + \frac{\epsilon}{16} + \mathcal{O}(\epsilon)^2, \quad \nu_{\parallel} = 1 + \frac{\epsilon}{12} + \mathcal{O}(\epsilon)^2, \quad \beta = 1 - \frac{\epsilon}{6} + \mathcal{O}(\epsilon)^2 \quad (6)$$

where $\epsilon = 4 - d$. As expected, the critical exponents do not depend on the value of the diffusion rate.

To get an insight how the diffusion rate D changes the critical creation rate λ_c , let us consider the field theoretic calculation at the beginning. Due to convergence problems upon applying dimensional regularization, one is forced to replace the parameter κ by mass $m^2 = \kappa - \kappa_c(D)$ where

$$\kappa_c = g^2 \int \frac{d^d q}{\frac{D}{2d} q^2 - \kappa_c} \quad \text{with } \sqrt{2d\kappa_c/D} < |q| < \Omega \quad (7)$$

is chosen in a way that m becomes zero at criticality [7]. The lower boundary is given by the fact, that for q smaller than $\sqrt{2d\kappa_c/D}$ the propagator becomes zero. The upper boundary Ω is a cutoff scale in momentum space and should be sent to infinity under continued renormalization. Inserting the d -dimensional surface element and expanding the right hand side of (7) in powers of q one obtains an integral over a geometric series

$$\kappa_c = \frac{4d\pi^{d/2}g^2}{D\Gamma(d/2)} \int_{\sqrt{2d\kappa_c/D}}^{\Omega} dq q^{d-3} \sum_{i=0}^{\infty} \left(\frac{2dq^{-2}\kappa_c}{D} \right)^i. \quad (8)$$

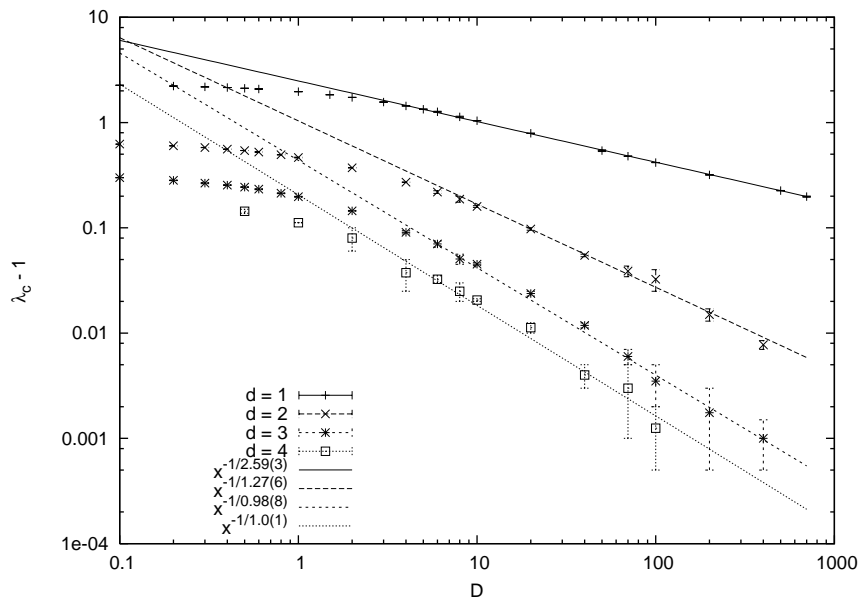


Figure 1. Log-Log plot of $\lambda_c(D)$ derived by Monte-Carlo simulation. The straight lines show power laws, fitted to the large D part of the numerical data.

By dimensional analysis one can show that for $d < 2$ only the lower boundary and for $d > 2$ only the upper boundary contributes to the integral. As we are interested in the asymptotic behavior for large D , we may approximate the sum by the leading term 1. Integrating the remaining part then yields the crossover behavior for different spatial dimension

$$\lambda_c(D) - 1 \sim \begin{cases} D^{-\frac{d}{4-d}} & \text{for } d < 2 \\ \frac{\log D}{D} & \text{for } d = 2 \\ D^{-1} & \text{for } d > 2 \end{cases} \quad (9)$$

These results compare very well to the predictions made in [10] but here they are obtained in a much simpler and direct way. From (9) we read off the crossover exponents

$$\phi(d) = \begin{cases} 3 & \text{for } d = 1 \\ 1 + \log. \text{ corrections} & \text{for } d = 2 \\ 1 & \text{for } d \geq 3 \end{cases} \quad (10)$$

3. Numerical results

In order to substantiate these analytical results, we simulated the diffusive contact process on periodic d -dimensional lattices using the following dynamic rules: Select a random lattice site and if there is a particle perform one of the following moves:

- (i) remove the selected particle with probability $1/(1 + D + \lambda)$,

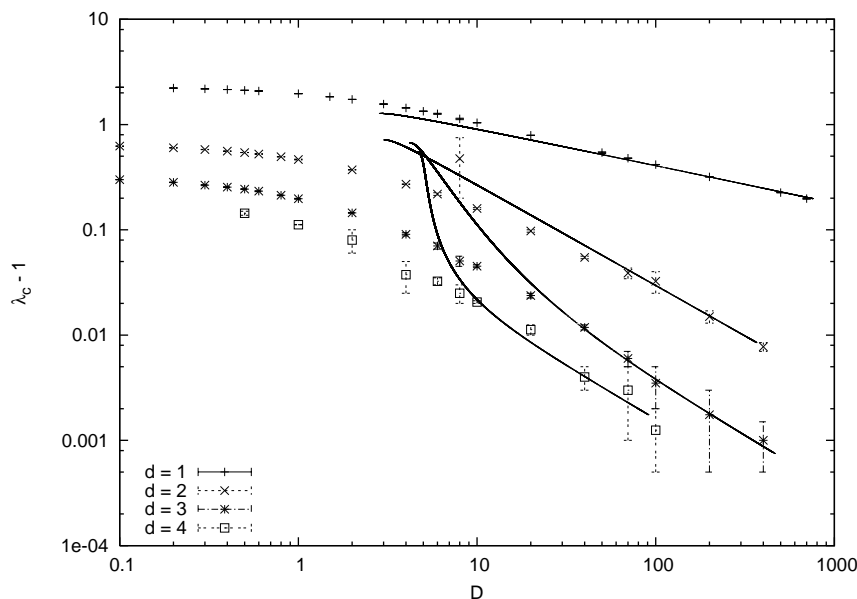


Figure 2. The same simulation results as in figure 1 compared to the numerically integrated predictions of eq. (7) (solid lines). Since the iteration becomes unstable for small diffusion rate, this region is not accessible by this approach.

- (ii) create a particle on a randomly chosen neighbouring site with prob. $\lambda / (1 + D + \lambda)$ if this site is empty, or
- (iii) move the particle to randomly selected neighbouring site with prob. $D / (1 + D + \lambda)$ if this site is empty.

For various values of the diffusion rate D we measured the particle density in a system starting with a fully occupied lattice and averaged over at least 20 independent runs. By varying the creation rate λ and searching for a power law decay we determined the critical threshold λ_c as a function of D . These critical lines for one, two, three and four spatial dimensions are shown in figure 1. The plots clearly indicate the expected power law behavior for large diffusion rates and the exponents are in good agreement with our analytical results. For small D , however, the observed deviations from a pure power law are not very surprising because the diffusion length becomes comparable to the lattice spacing.

So far we verified the predicted asymptotic behavior (9). In order to investigate the next-leading corrections of the geometric series, we integrate (8) numerically and compare it with the results of the simulations (figure 2). Although for small D this solution is plagued by numerical instabilities rendering the results unusable, the reader should notice that for spatial dimension $d = 1$ the numerical integration becomes almost exact over at least two decades and does not show the lift off behavior like in other dimensions.

4. Concluding remarks

In this paper we have presented a simple field-theoretic calculation and extensive numerical simulations in order to investigate how the critical parameter depend on the diffusion rate of a DP process in various dimensions. Our field-theoretical result confirms previous results by Konno [10], derived here in a much simpler way. Moreover we have presented numerical results which are in agreement with these predictions, refining previous results by Dantas *et al.* [11] and extending them to higher space dimensions up to $d = 4$. Especially in one spatial dimension, the field-theoretic prediction to one loop order, when evaluated numerically, coincides almost perfectly with the Monte Carlo estimates. This is surprising since one expects loop corrections to be more relevant in lower dimensions.

References

- [1] J. Marro and R. Dickman. *Nonequilibrium phase transitions in lattice models*. Cambridge University Press, Cambridge, UK, 1999.
- [2] W. Kinzel. Phase transitions of cellular automata. *Z. Phys. B*, 58:229, 1985.
- [3] H. Hinrichsen. Non-equilibrium critical phenomena and phase transitions into absorbing states. *Adv. Phys.*, 49:815, 2000. [cond-mat/0001070].
- [4] G. Ódor. Universality classes in nonequilibrium lattice systems. *Rev. Mod. Phys.*, 76:663, 2004.
- [5] S. Lübeck. Universal scaling behavior of non-equilibrium phase transitions. *Int. J. Mod. Phys. B*, 18:3977, 2004.
- [6] J. Cardy. *Scaling and renormalization in statistical physics*. Cambridge University Press, Cambridge, U.K., 1996.
- [7] Uwe Claus Täuber. Field theory approaches to nonequilibrium dynamics. *Lect. Notes Phys.*, 716:295, 2007.
- [8] H. Hinrichsen. On possible experimental realizations of directed percolation. *Braz. J. Phys.*, 30:69, 2000.
- [9] K. A. Takeuchi, M. Kuroda, H. Chaté, and M. Sano. Directed percolation criticality in turbulent liquid crystals. *Phys. Rev. Lett.*, 99:234503, 2007.
- [10] Norio Konno. Asymptotic behavior of basic contact process with rapid stirring. *J. Theo. Prob.*, 8:833, 1995.
- [11] W. G. Dantas, M. J. de Oliveira, and J. F. Stilck. Revisiting the one-dimensional diffusive contact process. *J. Stat. Mech.*, page P08009, 2007.
- [12] H. K. Janssen. On the non-equilibrium phase transition in reaction-diffusion systems with an absorbing stationary state. *Z. Phys. B*, 42:151, 1981.